Separation Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713708471

Separating Nonferrous Metals in Incinerator Residue Using Magnetic Fluids

S. E. Khalafalla[a]; G. W. Reimers[a]
[a] TWIN CITIES METALLURGY RESEARCH CENTER U.S. BUREAU OF MINES, TWIN CITIES, MINNESOTA

To cite this Article Khalafalla, S. E. and Reimers, G. W.(1973) 'Separating Nonferrous Metals in Incinerator Residue Using Magnetic Fluids', Separation Science and Technology, 8: 2, 161 — 178
To link to this Article: DOI: 10.1080/00372367308057994
URL: http://dx.doi.org/10.1080/00372367308057994

SEPARATION SCIENCE, 8(2), pp. 161–178, 1973

Separating Nonferrous Metals in Incinerator Residue Using Magnetic Fluids

S. E. KHALAFALLA* and G. W. REIMERS

TWIN CITIES METALLURGY RESEARCH CENTER
U.S. BUREAU OF MINES
TWIN CITIES, MINNESOTA 55111

Abstract

A magnetogravimetric separation method has been adopted to fractionate nonferrous metals from the residue of incinerated urban refuse. By properly placing a colloidal solution of magnetite in kerosene between the poles of a magnet, it was possible to separate particles according to their density spectrum in a continuous fashion. Screening the incinerator residue to −4, +14 mesh removed the bulk of silica (glass), reduced the sample size by about 52%, and gave a residue rich in aluminum, copper, and zinc with less than 3% silica. Separation of this residue into its components was accomplished using a permanent magnet and a magnetic fluid with a saturation magnetization of 80 G. The light fraction collector contained 58% of the sample and was rich in aluminum (84.7%). The heavy residue was further fractionated using an electromagnet with the field adjusted to 2.4 kOe and a magnetic fluid with a saturation magnetization of 215 G. Materials with mass density less than 7.2 g/cm^3 were received in the light fraction receptacle and were rich in zinc, while the heavy fraction receiver contained material rich in copper. The bulk of magnetic fluid that adhered to the separated particles was recovered by floating on water and skimming off with the aid of moving magnets.

INTRODUCTION

Magnetic fluids are Newtonian liquids that retain their fluidity in the presence of an external magnetic field. The fluids are ultrastable colloidal

* Author to whom correspondence should be sent. Present address: P.O. Box 1660, Twin Cities, Minnesota 55111.

suspensions of submicron-sized ferromagnetic or ferrimagnetic particles in liquid carriers such as hydrocarbons (kerosene), silicones, water, and fluorocarbons.

Since the discovery by Papell (*1*) that magnetic fluids can be lifted against gravity in magnetic fields, numerous applications of the transient zero-gravity conditions that can be developed within these fluids have appeared. Kaiser (*2*) used an on–off magnetic field for the separation of particles positioned at different elevations in the fluid according to density. Rosensweig (*3*) used the fluid as a horizontal sieve. Both methods utilized the magnetic levitation force created within the fluid in an inhomogenous magnetic field as an antigravity force; that is, the former force was always oriented in the vertical direction.

The present paper describes a new procedure for differential separation of materials according to their density spectrum. This was achieved by feeding the mixture through a cell containing a kerosene-base magnetic fluid having a moderate saturation magnetization of about 100 G. The cell was obliquely sandwiched between two steel blocks on the poles of a laboratory permanent magnet. It was sufficient to have an average magnetic field gradient of about 1.5 kOe/cm. By directing the magnetic levitation force in a nearly horizontal direction, the resultant vector summation of this force and the vertical gravitational force on each particle was in an oblique direction at an angle, θ, from the horizontal, which was a function of the particle density. Particles of various densities followed different trajectories in the fluid and were collected separately at different receptacles in the exit section. The concept in this method is radically different from that of previous procedures that utilize the magnetic levitation force in a vertical direction and hence have either to change the magnetic field gradient (*2*) by use of an intermittent variable electromagnet, or to use a bottomless container (*3*) for the magnetic fluid with the magnetomotive force holding the fluid in place.

First, the theory and principles underlying this new procedure for continuous particle separation will be developed. This method will then be adopted to fractionate nonferrous metals from the residue of an experimental solid waste incinerator.

EXPERIMENTAL WORK

Magnetic colloids used in this investigation were prepared in our laboratory. The colloid consisted of magnetite in the concentration range 10 to 30% dispersed in kerosene in the presence of 7 to 10% oleic acid as a protective colloid.

Samples of the nonferrous residue of an experimental solid waste incinerator were kindly supplied by M. J. Spendlove, Research Director, College Park Metallurgy Research Center, U.S. Bureau of Mines, College Park, Maryland.

RESULTS AND DISCUSSION

Some Properties of the Magnetic Fluid

Figure 1 is a photograph of a magnetic fluid sample, density about 0.8 g/ml, containing water as a contaminant and some alumina pellets having a density of 3.8 g/cm^3. Naturally, these constituents will sink below the magnetic fluid. Figure 2 illustrates the creation of a magnetic-levitation force, sometimes called an antigravity force, in a magnetic fluid-magnetic field combination. In this figure the same container shown in Fig. 1 was placed on a laboratory horseshoe magnet, whereby both water and alumina were floated on top of the fluid. Thus in a magnetic field the fluid appears to have a density greater than 3.8 g/ml. With fairly concentrated fluids and higher magnetic field gradients, platinum wires were levitated to the surface of a kerosene-base magnetic fluid, and the fluid thereby exhibited an apparent density of at least 21.5 g/ml.

The apparent increase of the density of these fluids with increasing magnetic field gradient suggests their utilization as substitutes for heavy liquids in mineral separation. The fact that their apparent density can be varied at will would enable the selective separation of particles according

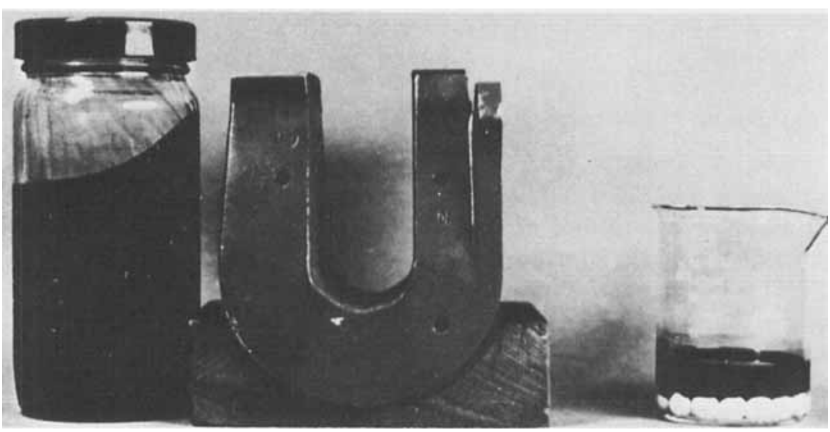

FIG. 1. Alumina beads and water submerged in magnetic fluid.

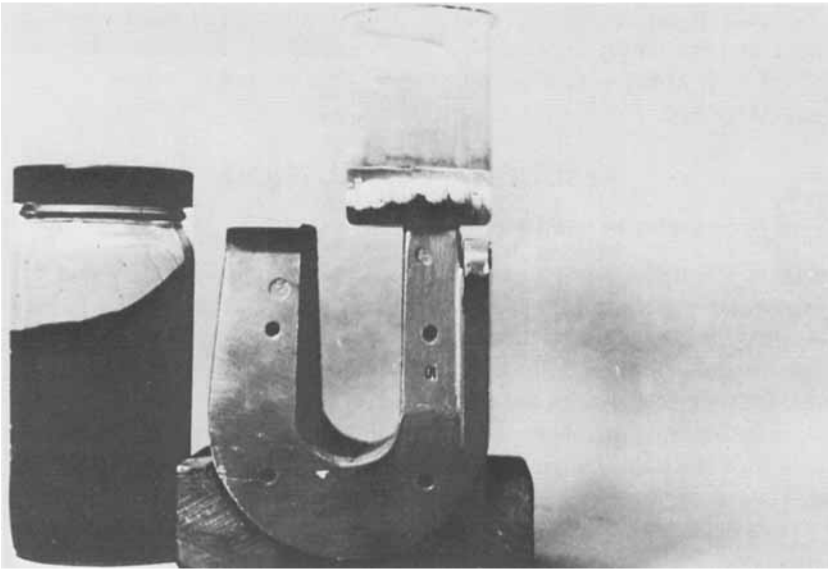

FIG. 2. Magnetic field gradient floats alumina beads and water on a magnetic fluid of lower density.

to their density spectrum by applying the principles of the Cartesian diver. Figure 3 shows a portion of magnetic fluid defying gravity under the influence of a laboratory permanent magnet. A unique property of these fluids is that when the magnetic particles are attracted to a magnet, the carrier fluid moves along with the particles so that the carrier fluid appears to be magnetic.

Continuous Magnetogravimetric Particle Separation

Figure 4 illustrates the principles of the newly developed magneto-gravimetric particle separation by nonvertical levitation forces. This approach allows the continuous separation of materials by means of a single magnetic fluid–magnetic field combination. The mixture is fed from a chute, and the components are received at different receptacles in the exit section. The fluid here acts as a density spectrograph, diverting particles in different trajectories according to their density. A major advantage of this technique is that magnetic particles are deflected to the backside of the separation cell and hence can be mechanically separated from the various nonmagnetic fractions. Magnetic particles cannot be simultane-

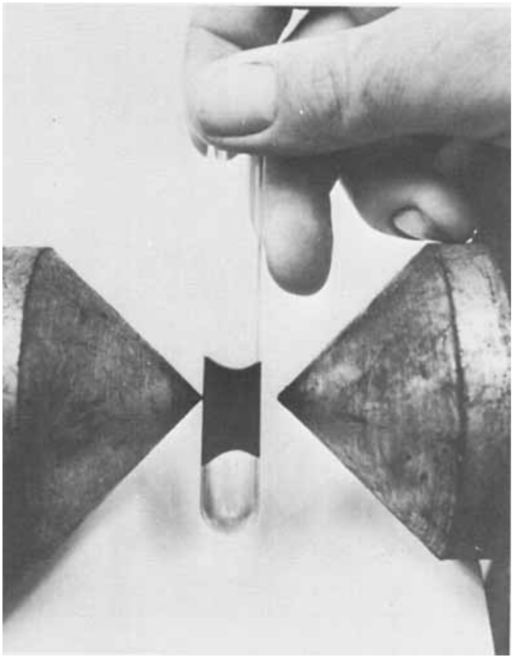

FIG. 3. Magnetic fluid in a test tube defying gravity when placed near the poles of a permanent magnet.

ously separated in the prior art of "float or sink" using magnetic fluids. In Fig. 4, a mixture of alumina balls (white) and lead shot (black) was poured into the magnetic fluid contained in a cell that was obliquely sandwiched between two steel blocks on the poles of a laboratory magnet. The mixture was thereby instantly separated, with the components falling separately into the two beakers in the left side of the picture. The white alumina was partially covered with the dark fluid in the front beaker. Because of its lower density, the adhering fluid can be recovered by washing with water.

To obtain the same separation with heavy liquids, a medium with specific gravity greater than 4 would be needed, whereby alumina would float and lead would sink. Such a liquid would be effective only for this separation; by contrast, the magnetic fluid can be worked to separate more components and to cover the entire density spectrum.

It should be noted that the horseshoe magnet in Fig. 4 was tilted on the table so that the magnetic field gradient lies off the vertical axis. The total

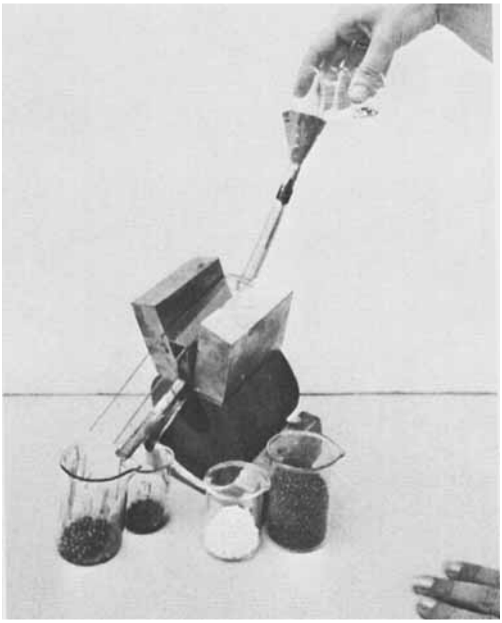

FIG. 4. Magnetogravimetric particle separator.

force acting on a body immersed in the magnetic fluid is the vector sum of the gravitational and levitational forces. Figure 5 shows a diagrammatic sketch of the orientation of these forces.

The magnitude of the gravitational force, $\mathfrak{f}_g$, on a body of density ρ' and volume V, immersed in a fluid of density ρ, is given by the Archimedes law as

$$\mathfrak{f}_g = V(\rho' - \rho)g \tag{1}$$

where g is the acceleration due to gravity.

The magnetic levitation force $\mathfrak{f}_l$ on the same body immersed in a magnetic fluid of magnetization M (magnetic moment per unit volume $\mu = M/4\pi$), placed in a magnetic field with gradient ∇H, is given (*4*) by

$$\mathfrak{f}_l = -\frac{1}{4\pi}\int_V M\nabla H\, dV = -\frac{V}{4\pi}\overline{M}\nabla H = -V\bar{\mu}\nabla H \tag{2}$$

where $\overline{M}$ is the volume-averaged magnitude of the vector $\mathbf{M}$, and μ is the volume-averaged fluid magnetic moment. The gravitational force is always

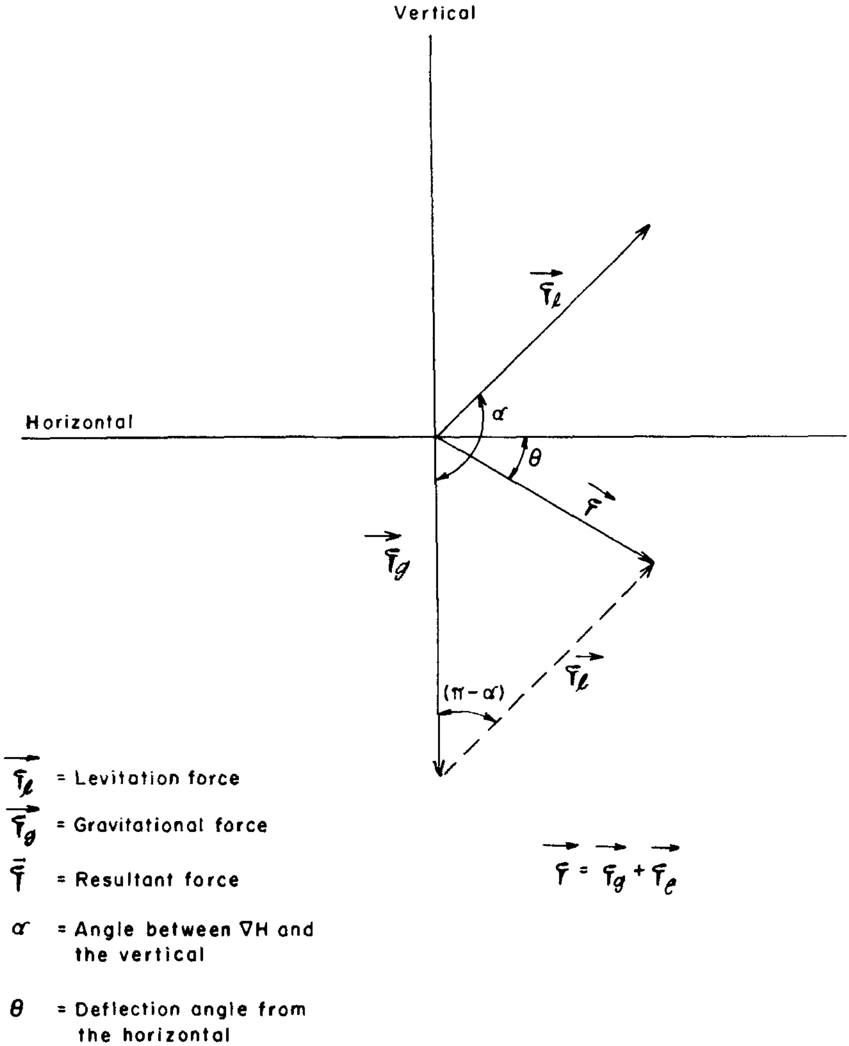

FIG. 5. Diagrammatic sketch of force orientations in material separation.

in the vertical direction (downwards), while the levitation force is in a parallel but opposite direction to that of ∇H.

In previous magnetic levitation experiments, ∇H was always in the vertical direction, and hence the net force acting (5) on a body immersed in a magnetic fluid would be

$$\mathfrak{f} = V\left[(\rho - \rho')g\boldsymbol{k} - \frac{1}{4\pi}\bar{M}\nabla H\right] \tag{3}$$

where $\boldsymbol{k}$ is a unit vector in the vertical direction.

In the new procedure, the vector describing ∇H is deliberately oriented at an angle α from the vertical. The resultant force $\mathfrak{f}$ acting on the immersed body will therefore deviate from the horizontal axis by an angle θ, as shown in Fig. 5. If a mixture of solid objects with different densities is placed in the fluid, the separate particles will be deflected through varying angles according to their densities. A heavy particle will have a large deflection angle θ from the horizontal, while a light object will have a smaller deflection angle. Separation becomes feasible with the addition of carefully placed slots and receptacles to collect and receive objects of the same density, or lying within a given density range.

Referring to Fig. 5, and assuming that the objects start in the magnetic fluid with a zero initial velocity, then

$$\frac{\sin(\pi - \alpha)}{\mathfrak{f}} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\mathfrak{f}_l}$$

or

$$\frac{\sin\alpha}{\mathfrak{f}} = \frac{\cos\theta}{\mathfrak{f}_l} \tag{4}$$

Hence the deflection angle is given by

$$\cos\theta = \frac{\mathfrak{f}_l \sin\alpha}{\mathfrak{f}}$$

$$= \frac{\mathfrak{f}_l \sin\alpha}{\{\mathfrak{f}_l^2 + \mathfrak{f}_g^2 + 2\mathfrak{f}_l\mathfrak{f}_g\cos\alpha\}^{\frac{1}{2}}} \tag{5}$$

where $\mathfrak{f}_g$ and $\mathfrak{f}_l$ are the magnitudes of the vectors defined in Eq. (1) and (2), respectively.

To establish the dependence of the deflection angle θ on the object's density, the trajectories of glass beads, aluminum chips, alumina pellets,

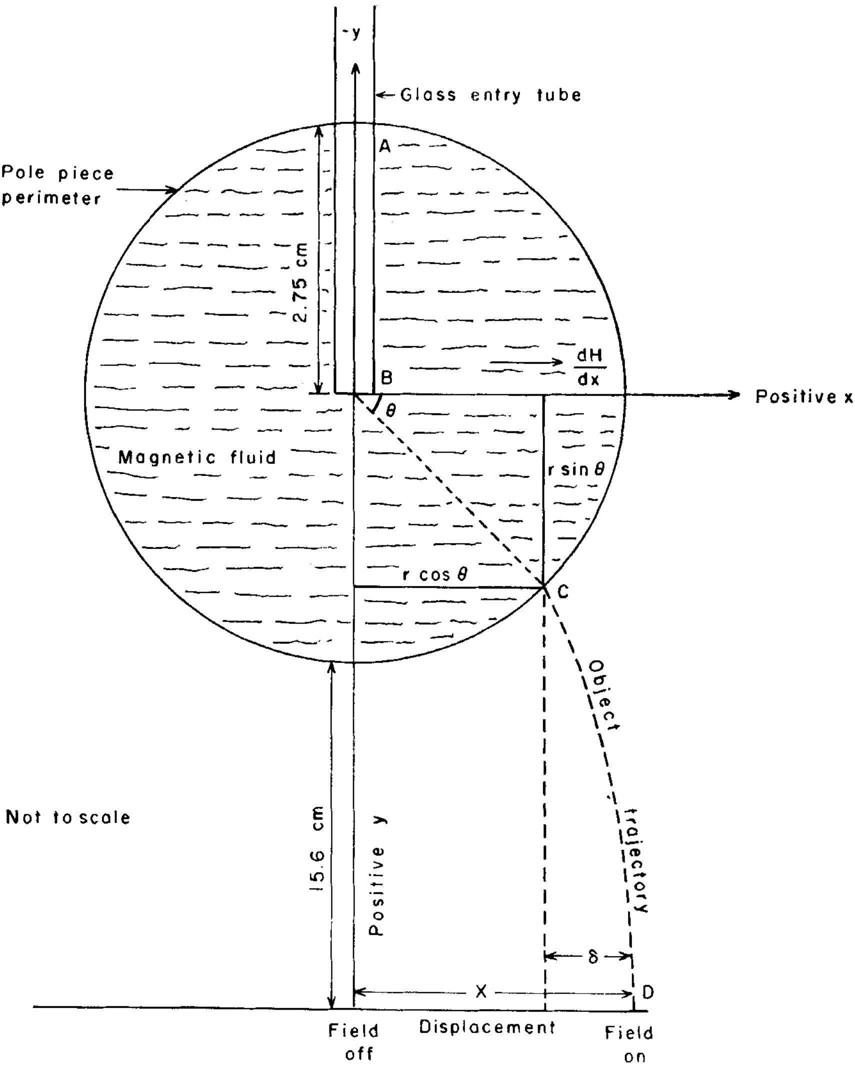

FIG. 6. Vertical cross-section of magnetic fluid cell between constant gradient pole tips of an electromagnet to determine horizontal displacement of objects with different densities.

and lead shots in a magnetic fluid were determined experimentally. Because of the difficulty in having zero initial velocity and kinetic energy for the particles submerged in the magnetic fluid, Eq. (5)—although qualitatively valid—cannot be subjected to exact experimental checking. Hence, an experiment was performed such that the levitation force was oriented along the horizontal direction, as shown in Fig. 6. In this experiment a magnetic fluid sample of density 0.96 g/ml and viscosity 2.72 cP was positioned between the constant gradient pole pieces of an electromagnet. The magnetic field $\boldsymbol{H}$ was mainly in the horizontal, x, direction, and the magnetic field gradient dH/dx was carefully measured as -0.3 kOe/cm. The average magnetic moment per unit volume of the fluid at the ambient magnetic field was 7.7 erg/Oe cm^3, which corresponds to an average fluid magnetization, $\overline{M}$, of 96.7 G. Objects with densities ranging from 2.5 to 11.3 g/cm^3 were fed through the magnetic fluid and received on a filter paper in a plane 15.6 cm below (18.4 cm from the center of the fluid). The impingment of a given object on the paper produced a spot, whose displacement X from the projection of the fluid center on the paper was carefully measured. The average displacements for the objects investigated, ± 1 standard deviation, are shown in the last column of Table 1.

Referring to Fig. 6, three regimes of particle motion must be considered:

I. First Regime

From Point A to Point B, the particle falls in the fluid under the influence of gravity alone. The equation of motion for a falling object in a viscous medium, as derived in the Appendix, is

$$v_y = gt - \frac{1}{2}g\left(\frac{b}{m}\right)t^2 + \frac{1}{6}g\left(\frac{b}{m}\right)^2 t^3 + \cdots \tag{6}$$

TABLE 1

Displacement of Objects According to Their Trajectories in a Magnetic Fluid

Sample	Density ρ (g/cm^3)	Volume V (cm^3)	Mass m (g)	Stokes correction factor b (g/sec)	Displacement X (cm)
Glass	2.50	0.067	0.169	0.128	4.87 ± 0.30
Aluminum	2.70	0.042	0.113	0.108	3.96 ± 0.37
Alumina	3.84	0.044	0.169	0.112	2.71 ± 0.30
Lead	11.34	0.024	0.272	0.102	0.57 ± 0.13

where v_y is the particle speed in the vertical (y) direction (positive y direction is downwards), b is the Stokes viscous drag factor, m is the mass of the body, and t is the elapsed time. The factor b is given by

$$b = 6\pi\eta r \tag{7}$$

where η is the viscosity coefficient of the fluid (2.72 cP in this experiment) and r is the particle radius.

Taking glass as an example, the vertical speed at Point B is given by

$$\begin{aligned} v_y(\mathrm{B}) &= gt_1 - \frac{1}{2}g\left(\frac{0.128}{0.169}\right)t_1^2 + \frac{1}{6}g\left(\frac{0.128}{0.169}\right)^2 t_1^3 - \cdots \\ &= 73.41 - 2.08 + 0.04 - \cdots \\ &\simeq 71 \text{ cm/sec} \end{aligned} \tag{8}$$

A first approximation for the elapsed time t_1 used in Eq. (8) was determined from the kinematic equation

$$S = \frac{1}{2}at_1^2 \quad \text{or} \quad t_1 = \left(\frac{2S}{a}\right)^{\frac{1}{2}} = \left(\frac{2S}{g}\right)^{\frac{1}{2}} \tag{9}$$

where S is the distance AB = 2.75 cm, through which the object actually fell, and a is the magnitude of the acceleration, equal to g, in the gravity field. The estimated value of t_1 amounted to 74.9 msec. Calculations in Eq. (8) show that even with glass beads, which exhibit the largest Stokes coefficient b, the frictional drag force is small compared to the gravitational force in these experiments. If this is true with glass, it would certainly be true with the other objects with smaller b coefficients (see fifth column of Table 1).

II. Second Regime

In the second region of motion, Point B to Point C (Fig. 6), the object is still in the fluid but is under the influence of both gravity and levitation forces. Acceleration due to the levitation force in the x-direction is given by Newton's second law as

$$a_x = \mathfrak{f}_l/m' \tag{10}$$

where m' is the effective mass of the object in the fiuid, given by

$$m' = V(\rho' - \rho) \tag{11}$$

Substituting for $\mathfrak{f}_l$ by its value from Eq. (2), one obtains

$$a_x = -\frac{V}{m'}\bar{\mu}\nabla H = -\bar{\mu}\frac{\nabla H}{(\rho' - \rho)} \tag{12}$$

The acceleration a_x is positive because ∇H is negative and ρ' is greater than ρ. The kinematic equation for motion in the x-direction with constant acceleration is

$$x = x(\mathrm{B}) + v_x(\mathrm{B}) + \frac{1}{2}a_x t^2 \tag{13}$$

Because Point B is taken as the origin of our coordinate system, $x(\mathrm{B}) = 0$ and $v_x(\mathrm{B}) = 0$. The horizontal distance x is related to the deflection angle θ by

$$x = r\cos\theta \tag{14}$$

Combining Eq. (12), (13), and (14), one obtains

$$r\cos\theta = \frac{1}{2}\bar{\mu}\frac{\nabla H}{(\rho - \rho')}t_2^{\,2} \tag{15}$$

where t_2 is the time taken by the object to travel from Point B to Point C in Fig. 6. This also equals the time taken by the object to move a distance ($r\cos\theta$) in the x-direction, or a distance ($r\sin\theta$) in the y-direction. The kinematic equation for motion in the vertical direction is

$$y(\mathrm{C}) = y(\mathrm{B}) + v_y(\mathrm{B})t_2 + \frac{1}{2}g t_2^{\,2} \tag{16}$$

Remembering that $y(\mathrm{C}) - y(\mathrm{B}) = r\sin\theta$, and that vertical speed $v_y(\mathrm{B})$ was calculated as 71 cm/sec in Eq. (8), a quadratic in t_2 may be derived from Eq. (16):

$$\frac{1}{2}g t_2^{\,2} + v_y(\mathrm{B})t_2 - r\sin\theta = 0 \tag{17}$$

and hence

$$t_2 = \frac{1}{g}[-v_y(\mathrm{B}) \pm \sqrt{v_y^2(\mathrm{B}) + 2rg\sin\theta}] \tag{18}$$

For a positive t_2, the root with the positive sign before the second term in the right-hand side of Eq. (18) is chosen. The dependence of the deflection angle θ on the density of the falling object is obtained by combining Eq. (15) and (18); thus

TABLE 2

Comparison of Theory with Experiment for Separation with Nonvertical Levitation Forces

Sample	Angle of deflection θ	Displacement X (cm)	
		Experimental	Calculated
Glass	74.5	4.87 ± 0.30	4.96
Aluminum	77.0	3.96 ± 0.37	4.23
Alumina	81.5	2.71 ± 0.30	2.70
Lead	88.0	0.57 ± 0.13	0.64

$$r \cos \theta = -\frac{1}{2}\frac{\bar{\mu}}{g^2}\frac{\nabla H}{(\rho - \rho')}[\sqrt{v_y^2(\mathrm{B}) + 2rg \sin \theta} - v_y(\mathrm{B})]^2 \tag{19}$$

Equation 19 cannot be solved explicitly for θ. The method of successive approximations and reiterations to consistency was performed with the aid of a computer and gave the values of θ shown in the second column of Table 2.

III. Third Regime

When the particle is ejected from the fluid at Point C, it travels the trajectory CD in air and impinges on the filter paper at Point D. The total linear deflection X from the vertical path (had there been no magnetic field) is given by

$$X = r \cos \theta + \delta \tag{20}$$

The deflection in air after emerging from Point C is given by

$$\delta = v_x(\mathrm{C})t_3 \tag{21}$$

where $v_x(\mathrm{C})$ is the particle speed in the x-direction at Point C, and t_3 is the time taken by the particle to fall a vertical distance of $(18.4 - r \sin \theta)$ in air.

For motion in the x-direction with a constant acceleration a_x,

$$v_x(\mathrm{C}) = \sqrt{2a_x(r \cos \theta)} \tag{22}$$

Substituting for the value of a_x from Eq. (12),

$$v_x(\mathrm{C}) = \left\{2\mu\frac{\nabla H}{(\rho - \rho')}(r \cos \theta)\right\}^{\frac{1}{2}} \tag{23}$$

From the kinematic equation of motion in the y-direction, the time of flight in air t_3 is given by

$$t_3 = \frac{1}{g}[-v_y(C) + \sqrt{v_y^2(C) + 2g(18.4 - r\sin\theta)}] \tag{24}$$

The vertical speed of the particle at Point C is given by

$$v_y(C) = \sqrt{v_y^2(B) + 2rg\sin\theta} \tag{25}$$

where $v_y(B)$ is calculated from Eq. (8) as 71 cm/sec for glass. Typical values of $v_{y(C)}$ are about 100 cm/sec. Substituting from Eqs. (21), (23), and (24) into Eq. (20), one obtains

$$X = r\cos\theta + \frac{1}{g}[\sqrt{v_y^2(C) + 2g(18.4 - r\sin\theta)} - v_y(C)] \cdot \sqrt{2\mu\frac{\nabla H}{(\rho - \rho')} r\cos\theta} \tag{26}$$

Inserting the values of θ obtained from the computer solution of Eq. (19) into Eqs. (25) and (26), one can calculate the expected displacement X for the objects studied. The results are given in the fourth column of Table 2. These theoretically predicted displacements agree reasonably well with the experimentally measured displacements. This agreement lends support to the theory of the new procedure of separating particles in a magnetic fluid according to their density spectrum.

Fractionation of Incinerator Residue

The magnetogravimetric separation technique with an obliquely oriented magnetic levitation force was adopted to fractionate the nonferrous metals from the residue of an experimental solid waste incinerator. The sample as received (−4, +20 mesh) was first passed through an Eriez Magnetics drum separator. (Reference to trade names is made to facilitate understanding and does not imply endorsement by the Bureau of Mines.) This step was necessary because the separator used in the following experiments lacked a means for continually removing magnetic particles. About 10% by weight of the sample was found to be magnetic. A number of trials were made at separating the −4 mesh material to gain experience in the separation procedure. It was found that screening the residue to −4, +14 mesh removed the bulk of silica (glass) from the metallic residue. This reduced the sample size by about 52% and gave a residue rich in aluminum, copper, and zinc with less than 3% silica.

Separation of aluminum from 1500 g of the metallic (−4, +14 mesh) residue was accomplished using a permanent magnet and a magnetic fluid with a saturation magnetization of 80 G. The magnet was positioned to levitate material of specific gravity less than 4.0 into the light portion collector. As a result, 58% by weight of the sample was collected as "lights." A vibratory feeder supplied a continuous flow of residue into the separator where an instantaneous split was made. The two components of the split were then diverted into water-filled collectors. The bulk of magnetic fluid retained on the particles, being lighter than water, floated to the surface for return to the separator. The particles were further cleaned by sweeping adhering magnetic fluid up into the water with a small magnet. Figure 7 is a photograph of the separator used.

Zinc was separated from the heavy portion of the sample using an electromagnet with the field adjusted to 2.4 kOe, a 1-in. pole piece gap, and a magnetic fluid with a saturation magnetization of 215 G. This setting of the electromagnet caused material having a specific gravity of 7.2 to be levitated into the lighter or zinc fraction collector. The zinc

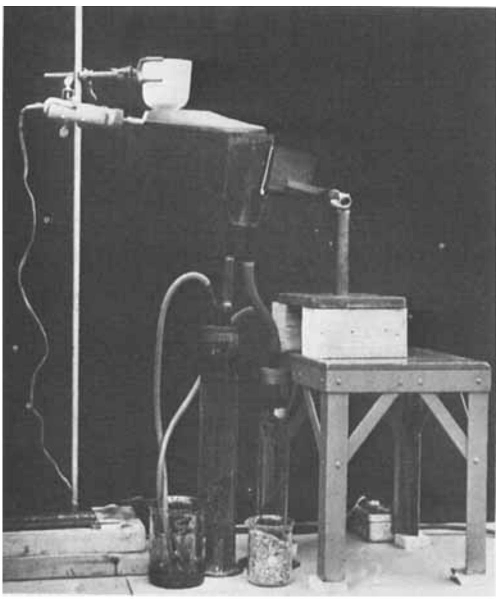

FIG. 7. Bench scale equipment for separation of Al from Cu and Zn in incinerator residue.

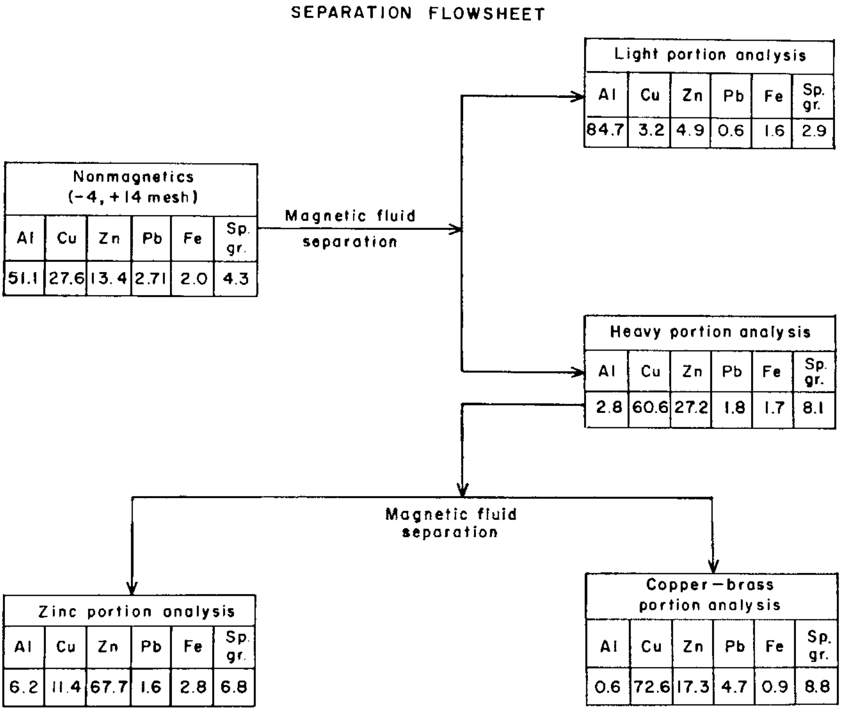

FIG. 8. Chemical analysis of incinerator residue fractions.

portion of this separation contained 20% by weight of the starting material.

Figure 8 shows a flow diagram of the separation method. The chemical analyses of the five prominent metals in the residue and the density of the various fractions in grams per cubic centimeter are shown in this figure.

A problem was initially encountered in making a good separation of the heavier material. Apparently as the sample dropped through the fluid into the separation zone, there was a velocity difference between the flat and prill-shaped particles which hampered separation. Moving the exit of the particle feeder close to the zone of levitation improved separation.

SUMMARY AND CONCLUSION

A laboratory technique for continuous magnetogravimetric separation of nonmagnetic particles according to their density was developed and tested. The method depends on the creation of a nonvertical levitation force in a magnetic fluid–magnetic field combination. The theory underlying the separation method was developed to relate the angular deflection of a particle trajectory within the fluid to the particle's density. The close agreement between experimental results and theoretical predictions suggests that the separation technique can be turned inside out for accurate determination of particle density. Particle constituency can also be determined from the measured displacement on a plane surface following the particle ejection from the fluid.

The new separation method was adopted to reclaim metal values from the residue of incinerated urban refuse. The same principles can be used to: (a) reclaim nonferrous metals from concentrated unburned urban refuse; (b) recover titanium, lead, and unalloyed copper and zinc from wasted ferrous scrap; (c) separate the gangues from native copper ores to upgrade their copper content; and (d) recover gold and precious metals from electronic and solder scrap. Separation processes based on the technique developed in this paper are covered by the Department of Interior, U.S. Patent Application Serial No. 248, 705.

APPENDIX

Kinematic Equation of Motion for a Falling Body in a Viscous Medium

The frictional force on a spherical body of radius r moving in a fluid of viscosity η is given by Stoke's law as

$$\mathfrak{f}_{f_r} = -6\pi\eta r\mathbf{v} = -b\mathbf{v} \tag{A-1}$$

where $\mathbf{v}$ is the terminal velocity of the particle, and b is the Stoke's coefficient.

Combining the magnitude of the frictional force with the gravitational force, $\mathfrak{f}_g = mg$, one obtains for the net vertical force f acting on the body

$$\mathfrak{f} = mg - bv = m\frac{dv}{dt}$$

$$\therefore \quad \frac{dv}{mg - bv} = \frac{1}{m}dt \tag{A-2}$$

Integrating Eq. (A-2) between the speeds v_0 and v, and times $= 0$ and t, one obtains

$$-\frac{1}{b}\left|\log(mg - bv)\right|_{v_0}^{v} = \frac{t}{m}$$

or

$$\frac{mg - bv}{mg - bv_0} = e^{-bt/m} \tag{A-3}$$

The terminal speed v would be given by

$$v = \frac{1}{b}[mg - (mg - bv_0)e^{-bt/m}]$$

$$= \frac{mg}{b}(1 - e^{-bt/m}) + v_0 e^{-bt/m} \tag{A-4}$$

Expanding the exponentials in Eq. (A-4), one obtains

$$v = \frac{mg}{b}\left[\frac{bt}{m} - \frac{1}{2}\left(\frac{bt}{m}\right)^2 + \left(\frac{1}{6}\frac{bt}{m}\right) - \cdots\right]$$

$$+ v_0\left[1 - \left(\frac{bt}{m}\right) + \frac{1}{2}\left(\frac{bt}{m}\right)^2 - \cdots\right] \tag{A-5}$$

And for an initial speed v_0 of zero,

$$v = gt - \frac{1}{2}g\left(\frac{b}{m}\right)t^2 + \frac{1}{6}g\left(\frac{b}{m}\right)t^3 - \cdots \tag{A-6}$$

REFERENCES

1. S. S. Papell and O. C. Faber, NASA TN D-3288, (February 1966), also U.S. Patent 3,215,572 (November 1965).
2. R. Kaiser, U.S. Patent 3,483,968 (December 1969).
3. R. E. Rosensweig, U.S. Patent 3,483,969 (December 1969).
4. R. E. Rosensweig, *AIAA J.*, *4*(10), 1751–1758 (1966).
5. S. E. Khalafalla and G. W. Reimers, 101st Meeting of the American Institute of Mining, Metallurgical, and Petroleum Engineers, San Francisco, California, February 1972, Preprint No. 72-B-63.

Received by editor July 19, 1972